CNLS-97-04

The Dynamics of Classical Chiral QCD_2 Currents

Robert de Mello Koch and João P. Rodrigues

Physics Department and Centre for Nonlinear Studies
University of the Witwatersrand
Wits 2050, South Africa

In this paper the dynamics of the classical chiral QCD_2 currents is studied. We describe how the dynamics of the theory can be summarized in an equation of the Lax form, thereby demonstrating the existence of an infinite set of conserved quantities. Next, the r matrix of a fundamental Poisson relation is obtained and used to demonstrate that the conserved charges Poisson commute. An underlying diffeomorphism symmetry of the equations of motion which is not a symmetry of the action is used to provide a geometric interpretation for the case of gauge group $SU(2)$. This enables us to show that the solutions to the classical equations of motion can be identified with a large class of curves, to demonstrate an auto-Bäcklund transformation and to demonstrate a non linear superposition principle. A link between the spectral problem for QCD_2 and the solution to the closed curve problem is also demonstrated. We then go on to provide a systematic inverse scattering treatment. This formalism is used to obtain the reflectionless single boundstate eigenvalue soliton solution.

August, 1997

1. Introduction.

QCD_2 with massless quarks in the fundamental representation of the gauge group has received considerable attention recently. This has occurred for essentially two reasons: the first is the claim using non-abelian bosonization that the theory should be integrable [1] ; the second stems from the observation [2] that for massless quarks in a representation with dimension larger than or equal to the dimension of the representation of a heavy probe charge, the theory exhibits screening.

In $1+1$ dimensions, integrability is usually associated with the existence of a large number of soliton-like solutions of the classical equations of motion which are recast in a Lax form, from which the existence of an infinite number of conservation laws follows. If these survive quantization, they normally imply an elastic, computable S-matrix [3] with no particle production.

There is, of course, considerable evidence of particle production [4] [5] in the large N_c expansion of QCD_2 [6] . It has been argued recently that this is not necessarily inconsistent with the existence of an integrable sector in the theory [7] . The issue of screening versus confinement for massless quarks in the fundamental representation of SU(N) has also been discussed recently [8] .

In a previous communication [9] , we analyzed the classical equations of motion of the theory in the light cone gauge $A_- = 0$ and established that the once integrated equation of motion for the currents is of the Lax form, demonstrating the existence of a infinite number of conserved quantities. We discovered that the equations of motion possess a continuous symmetry (diffeomorphism invariance in the x^- coordinate) which is not a symmetry of the action [1]. Making use of this symmetry and specializing to SU(2), we established that all classical Frenet-Serret curves with a constant term in the torsion are solutions of the equations of motion, and constructed a single soliton solution.

The original motivation of [9] was to use these classical solutions to construct bilinears [11] relevant to semiclassical approximations of fermionic theories. This has been implemented for the Gross-Neveu model [12] and the Thirring model [13] .

In this article, we are only interested in exploiting further the rich mathematical structure that emerges in the study of the equations of motion of the classical currents of the model. Specifically, in the gauge $A_- = 0$ these take the form

[1] Recently, a similar symmetry has been noted in IIB supergravity[10] .

$$\partial_+\partial_-^2\sigma = \frac{ig^2}{\sqrt{2}}[\partial_-^2\sigma, \sigma]. \tag{1.1}$$

where

$$\partial_-^2\sigma_{ab} = 2j_{ab}, \tag{1.2}$$

and j_{ab} is an SU(N) current[2].

In section 2, we show how equation (1.1) is arrived at and point out its Lax form. Following Fadeev, we construct the fundamental Poisson relation and recast the once integrated equation of motion as a zero curvature condition. In section 3, the chiral diffeomorphism in the x^- coordinate is exhibited and for $SU(2)$ it is used to map the most general solution of the equations of motion to appropriate Frenet-Serret curves. Using Darboux coordinates a second zero curvature condition is obtained. The Auto-Backlund transformation is identified and is used to obtain the single soliton solution of [9] . This Auto-Backlund transformation is then used to construct a non-linear superposition principle. In particular, we show how the full set of soliton solutions of the Sine-Gordon model are a subset of solutions generated by this non-linear superposition principle. For these solitons, we show further the the fermion number is quantized. The spectral problem of the zero curvature condition is found to be closely related to the spectral problem used to solve the closed curve problem. In section 4 our single soliton solution is shown to be the reflectionless single bound state eigenvalue inverse scattering solution. This construction is straightforwardly generalizable to $SU(N)$.

2. Some general observations.

We work in light cone co-ordinates[3], in the axial gauge $A_- = 0$. In terms of the quark fields $\psi_a = \left[\psi_{-a}, \psi_{+a}\right]^T$ which are in the fundamental representation of $SU(N)$ $(a = 1, 2, ...N)$, the Lagrangian density is given by

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}Tr(\partial_- A_+)^2 + i\sqrt{2}\psi_{+a}{}^\dagger(x)\partial_-\psi_{+a}(x) + \sqrt{2}\psi_{-a}{}^\dagger(x)(i\delta_{ab}\partial_+ + aA_{+ab})\psi_{-b}(x)\\ &- m(\psi_{-a}{}^\dagger\psi_{+a} + \psi_{+a}{}^\dagger\psi_{-a}).\end{aligned} \tag{2.1}$$

[2] These equations of motion also arise in certain compactifications of M(atrix) theory[14] .

[3] Our conventions are $x^\pm = \frac{1}{\sqrt{2}}(t \pm x)$

The equations of motion for both A_+ and ψ_+ do not involve derivatives with respect to x^+ and thus are interpreted as constraints. As usual, the constraints may be solved, leaving an effective theory for the quarks ($x = (x^+, x^-)$):

$$
\begin{aligned}
\mathcal{L} &= i\sqrt{2}\psi_a{}^\dagger(x)\partial_+\psi_a(x) + \frac{g^2}{2}\int dy^- \times \\
&\times \psi_a{}^\dagger(x)\psi_b(x)|x^- - y^-|\psi_b{}^\dagger(x^+, y^-)\psi_a(x^+, y^-) - \frac{g^2}{2N}\int dy^- \times \\
&\times \psi_a{}^\dagger(x)\psi_a(x)|x^- - y^-|\psi_b{}^\dagger(x^+, y^-)\psi_b(x^+, y^-).
\end{aligned}
\tag{2.2}
$$

where we have let $m \to 0$ and $\psi_{-a} \to \psi_a$. The dynamics of the classical system now follows from this Lagrangian in the usual way

$$
i\partial_+\psi_a(x) = \frac{g^2}{\sqrt{2}}\sigma_{ab}(x)\psi_b(x) \tag{2.3}
$$

where

$$
\sigma_{ab}(x) = \int dy^-|x^- - y^-|\big[\psi_a(y^-)\psi_b^\dagger(y^-) - \frac{1}{N}\delta_{ab}\psi_c(x^+, y^-)\psi_c{}^\dagger(x^+, y^-)\big]. \tag{2.4}
$$

In this last expression, all fields are evaluated at the same point in "time" x^+. In the above equations, the ψ fields are normal commuting functions. Note that from the definition (2.4) we find

$$
\partial_-^2\sigma_{ab}(x) = 2\Big(\psi_a(x)\psi_b^\dagger(x) - \frac{1}{N}\delta_{ab}\psi_c(x^+, y^-)\psi_c{}^\dagger(x^+, y^-)\Big). \tag{2.5}
$$

It is useful to note that the classical equations of motion (2.3) and (2.4) arise as the classical dynamics of the Hamiltonian

$$
H = -\frac{g^2}{4}\int d^2x Tr(\partial_-^2\sigma)\sigma, \tag{2.6}
$$

together with the (equal x^+) Poisson brackets

$$
\{\psi_a(x), \psi_b^\dagger(y)\} = \frac{i}{\sqrt{2}}\delta_{ab}\delta(x - y). \tag{2.7}
$$

An explicit realization for the Poisson brackets is

$$
\{\alpha, \beta\} = \frac{i}{\sqrt{2}}\int dx \sum_a \Big(\frac{\delta\alpha}{\delta\psi_a(x)}\frac{\delta\beta}{\delta\psi_a^\dagger(x)} - \frac{\delta\beta}{\delta\psi_a(x)}\frac{\delta\alpha}{\delta\psi_a^\dagger(x)}\Big). \tag{2.8}
$$

2.1. Lax Representation and Conservation Laws.

Using the equations of motion, the time dependence of $\psi_a\psi_b^\dagger$ is computed as

$$\partial_+(\psi_a\psi_b^\dagger) = \frac{ig^2}{\sqrt{2}}[\psi_a\psi_c^\dagger\sigma_{cb} - \sigma_{ac}\psi_c\psi_b^\dagger] \tag{2.9}$$

which may be written in the form of a Lax equation

$$\partial_+\partial_-^2\sigma = \frac{ig^2}{\sqrt{2}}[\partial_-^2\sigma, \sigma]. \tag{2.10}$$

This last equation is nothing but the conservation of the probability 2-current. However, due to the fact that the trace of the $-$ component of the probability current vanishes, the trace of any power of the $+$ component of the probability current is conserved (with respect to the "time" x^+). Since this quantity is local, it furnishes an infinite number of conserved quantities, indexed by the continous label x^-. Of course, this conservation law arises from the usual global $U(1)$ symmetry. What is suprising however, is that in this case, it is not the total charge (i.e. the integrated charge density) that is conserved, but rather the charge density itself. Note the close similarity between (2.10) and the dynamical equation for the Heisenberg spin chain

$$\partial_+ S = ig^2[\partial_-^2 S, S], \tag{2.11}$$

where S may be expanded in the basis of generators of $SU(2)$. A close connection between the Heisenberg spin chain and $SU(2)$ chiral QCD_2 will emerge when we perform a systematic inverse scattering treatment in a later section.

Consider next the time dependance of $\partial_-\sigma_{ab}(x)$. From the definition (2.4)

$$\begin{aligned}\partial_+\partial_-\sigma_{ab}(x) &= \partial_+\int dy^-\epsilon(x^- - y^-)\psi_a(y^-)\psi_b^\dagger(y^-)\\ &= \frac{ig^2}{\sqrt{2}}\int dy^-\epsilon(x^- - y^-)[\partial_-^2\sigma, \sigma]_{ab}.\end{aligned} \tag{2.12}$$

Now, noticing that

$$\partial_-[\partial_-\sigma(x), \sigma(x)] = [\partial_-^2\sigma(x), \sigma(x)] \tag{2.13}$$

we find, after integrating by parts and dropping the boundary term

$$\partial_+\partial_-\sigma_{ab}(x) = \frac{ig^2}{\sqrt{2}}[\partial_-\sigma(x), \sigma(x)]_{ab} \tag{2.14}$$

which is again of the Lax form. This Lax pair again gives an infinite number of conserved quatities contained in the conservation law

$$\partial_+ Tr\Big(\partial_-\sigma(x)\Big)^n = 0 \tag{2.15}$$

The conserved charges discussed above are all gauge invariant quantities. Note that the equations (2.14) and (2.10) are invariant under diffeomorphisms of the form $x^- \to f(x^-)$ [9] . The action however is not invariant under this symmetry. This diffeomorphism invariance will be used in the next section to provide a geometric interpretation of the problem.

Finally we comment that it is also a simple task to show that

$$\partial_+ Tr\Big(\sigma^n(x)\Big) = 0 \tag{2.16}$$

which again represents an infinite number of conserved quantities. The conserved quantities (2.15) and (2.16) are both non-local functionals of the original field variables. For a discussion on non-local conservation laws for the Heisenberg magnet, see [15] .

2.2. The r matrix.

The existence of an infinite sequence of conservation laws does not automatically guarantee integrability of the classical dynamics. One must prove in addition, that these conserved quantities Poisson commute. A very elegant way in which this is demonstrated is due to Faddeev [16] . The idea is to introduce the r matrix, defined by the fundamental Poisson relation (FPR)

$$\{U(x,\lambda) \overset{\otimes}{,} U(y,\mu)\} = \big[r(\lambda-\mu), U(x,\lambda)\otimes I + I\otimes U(y,\lambda)\big]\delta(x-y), \tag{2.17}$$

where we have introduced the notation

$$\{A \overset{\otimes}{,} B\}_{ik|lm} = \{A_{il}, B_{km}\}. \tag{2.18}$$

Usually the FPR is used to prove that certain conserved monodromies Poisson commute. The treatment for chiral QCD_2 is much simpler. Considering $U(x,\lambda) = \frac{i}{2\lambda}\partial_-^2\sigma(x)$, and taking traces of the above equation immediatly implies that

$$\{Tr(\partial_-^2\sigma)(x), Tr(\partial_-^2\sigma)(y)\} = 0. \tag{2.19}$$

Thus, if we can find an r matrix such that (2.17) holds, then we have demonstrated that our classically conserved quantities Poisson commute. For the case of the gauge group $SU(2)$, it is not difficult to verify that $\partial_-^2\sigma(x)$ obeys exactly the same Poisson bracket relations as the Heisenberg spin chain variables $\sigma^a S^a(x)$. In addition, the r matrix for the Heisenberg spin chain, where $U(x,\lambda) = \frac{i}{2\lambda}\sigma^a S^a(x)$ is known. Thus, the r matrix defined by (2.17) is nothing but the r matrix of the Heisenberg spin chain! In a natural basis, the r matrix is given by $r = P/(\lambda-\mu)$ [16] where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.20}$$

2.3. Zero Curvature Condition.

We begin by noting that the integrability condition of the linear equations

$$\partial_-\sigma_{ab}(x)\phi_b(x) = \alpha\phi_a(x) \tag{2.21}$$

$$\partial_+\phi_a = -\frac{ig^2}{\sqrt{2}}\sigma_{ab}(x)\phi_b(x) \tag{2.22}$$

reproduces (2.14) . Now, shifting $x^- \to x^- + \eta$ and using first order perturbation theory to compute $\delta\phi = \eta\partial_-\phi$ we are led to

$$\partial_-\phi_a = -\frac{ig^2}{\sqrt{2}}x^+\partial_-\sigma_{ab}(x)\phi_b. \tag{2.23}$$

Clearly, the pair of equations (2.23) and (2.22) , which constitute the linear problem, also have the diffeomorphism symmetry. The integrability condition of (2.23) and (2.22) now provide a zero curvature representation

$$[\partial_+ + \frac{ig^2}{\sqrt{2}}\sigma, \partial_- + \frac{ig^2}{\sqrt{2}}x^+\partial_-\sigma] = 0 \tag{2.24}$$

of (2.14) . This zero curvature representation is well suited to a systematic inverse scattering treatment, which is provided in section 4.

3. The case of SU(2): Some Preliminary Comments.

In this section we specialize our discussion to the case $N = 2$.

3.1. The Geometry of the Lax Pair.

Expanding the Lax equation (2.14) in the basis provided by the Pauli matrices (σ^i) as

$$\sigma_{ab}(x) = \frac{1}{2} f^i(x) \sigma^i_{ab} \tag{3.1}$$

yields

$$\partial_+ \partial_- f^i = -\frac{g^2}{\sqrt{2}} \epsilon^{ijk} \partial_- f^j f^k \tag{3.2}$$

After using the diffeomorphism symmetry to change to the new variable $s(x^-)$, such that $\partial_s f^i \partial_s f^i = 1$, we are able to interpret s as the arclength of the curve traced out by the three co-ordinates (f^1, f^2, f^3) at any given time x^+. Given the arclength and position of the curve it is a simple matter to compute the explicit expressions

$$(e_1)^i = \partial_s f^i \tag{3.3}$$

$$(e_2)^i = \frac{1}{\kappa} \partial_s^2 f^i \tag{3.4}$$

$$(e_3)^i = \frac{1}{\kappa} \epsilon^{ijk} \partial_s f^j \partial_s^2 f^k, \tag{3.5}$$

for the normal, binormal and osculating normal and

$$\kappa = \sqrt{\partial_s^2 f^i \partial_s^2 f^i} \tag{3.6}$$

$$\tau = \frac{1}{\kappa^2} \epsilon^{ijk} \partial_s f^i \partial_s^2 f^j \partial_s^3 f^k, \tag{3.7}$$

for the curvature and the torsion. The dynamics of the three unit normals e_i is easily found, upon using (3.2)

$$\partial_+ (e_n)_i = -\sqrt{2} g^2 \epsilon^{ijk} f^j (e_n)_k \tag{3.8}$$

In addition to this, the propagation of the curve in arclength is described by the Frenet-Serret equations which read

$$\partial_s e_1 = \kappa e_2 \tag{3.9}$$

$$\partial_s e_2 = \tau e_3 - \kappa e_1 \tag{3.10}$$

$$\partial_s e_3 = -\tau e_2. \tag{3.11}$$

3.2. A Second Zero Curvature Condition.

Now, transforming to Darboux co-ordinates

$$z_l = \frac{(e_2)_l + i(e_3)_l}{1-(e_1)_l} \quad (e_1)_l^2 + (e_2)_l^2 + (e_3)_l^2 = 1 \tag{3.12}$$

we are led to the pair of Riccati equations

$$\partial_- z_l = -i\tau z_l + \frac{\kappa}{2}(z_l^2 + 1) \tag{3.13}$$

$$\partial_+ z_l = -iaz_l + \frac{1}{2}(b+ic)z_l^2 + \frac{1}{2}(b-ic), \tag{3.14}$$

where

$$a = -if^i \partial_s f^i, \tag{3.15}$$

$$b = \frac{1}{\kappa\tau}\left[f^i\partial_s^3 f^i - \frac{1}{\kappa^2}\partial_s^2 f^j \partial_s^3 f^j f^i \partial_s^2 f^i + \kappa^2 f^i \partial_s f^i\right] \tag{3.16}$$

and

$$c = \frac{1}{\kappa} f^i \partial_s^2 f^i. \tag{3.17}$$

The Riccati equations derived above, and their relation to the Frenet frame will prove to be vital for the construction of auto-Bäcklund transformations. It is simple to verify that this is simply the coset space representation $z = \frac{v_1}{v_2}$ of the following pair of linear eigenvalue problems

$$\begin{bmatrix} \partial_s \beta_1 \\ \partial_s \beta_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{2}\tau & -\frac{\kappa}{2} \\ \frac{\kappa}{2} & \frac{-i}{2}\tau \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = U_{ij}\beta_j \tag{3.18}$$

$$\begin{bmatrix} \partial_+ \beta_1 \\ \partial_+ \beta_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{2}a & -\frac{1}{2}(c-ib) \\ \frac{1}{2}(c-ib) & \frac{-i}{2}a \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = V_{ij}\beta_j. \tag{3.19}$$

The integrability condition for this pair of equations implies[4]

$$\partial_+\tau = \sqrt{2}g^2 \quad \partial_+\kappa = 0. \tag{3.20}$$

This last result is remarkable and unexpected. It implies that *any* curve may be mapped into a solution of QCD_2.

The connection between soliton solutions and curves has been discussed by a number of authors [17] . In particular, Lamb [18] shows how the single soliton solution of the Sine-Gordon model can be related to a curve of curvature $\kappa = asech(as)$ and torsion $\tau =$ constant. This curve, upon identifying $\tau = \sqrt{2}g^2x^+$, provides a soliton solution of the equation (3.2) . Explicitely, we have

$$\begin{aligned} f_1 &= \frac{2a}{2g^4x^{+2}+a^2} sech(as) sin(\sqrt{2}g^2sx^+) \\ f_2 &= \frac{2a}{2g^4x^{+2}+a^2} sech(as) cos(\sqrt{2}g^2sx^+) \\ f_3 &= s - \frac{2a}{2g^4x^{+2}+a^2} tanh(as). \end{aligned} \tag{3.21}$$

It is a straightforward excercise to explicitly verify that these f_i do indeed satisfy the equation of motion (3.2) . We also state the solution with constant curvature κ and torsion $\tau = \frac{g^2}{\sqrt{2}}x^+$

$$\begin{aligned} f_1 &= \frac{2\kappa}{g^4x^{+2}+2\kappa^2} sin(\sqrt{\kappa^2+\frac{g^4}{2}x^{+2}}s) \\ f_2 &= \frac{2\kappa}{g^4x^{+2}+2\kappa^2} cos(\sqrt{\kappa^2+\frac{g^4}{2}x^{+2}}s) \\ f_3 &= -\frac{g^2x^+s}{\sqrt{g^4x^{+2}+2\kappa^2}}. \end{aligned} \tag{3.22}$$

[4] These restrictions on the curvature and the torsion can also be obtained directly as the integrability condition for the equations of motion (3.2) and the Frenet-Serret equations [9] .

3.3. An Auto-Bäcklund Transformation.

In this section, we will concentrate our attention on the class of constant torsion curves. Of course, this is only a small subset of all possible solutions to our equations of motion, but as we shall see, it includes a number of interesting solutions.

The geometrical interpretation of a Bäcklund transformation is as a transformation that takes a given pseudospherical surface[5] to a new pseudospherical surface. Moreover, a Bäcklund transformation takes asymptotic lines to asymptotic lines. The fact that asymptotic lines on a pseudospherical surface have constant torsion, is the first hint that it is possible to restrict the Bäcklund transformation to get a transformation that carries constant torsion curves to constant torsion curves. That this is indeed possible is the result of a theorem due to Calini and Ivey[19] : Assume that the osculating normal(e_3), binormal (e_2) and tangent (e_1) vectors to a constant torsion (τ) curve, together with the curves curvature (κ) are known. The curve with Frenet frame given by

$$e_1' = e_1 + (1 - cos\theta)sin\beta(cos\beta e_2 - sin\beta e_1) + sin\theta sin\beta e_3, \tag{3.23}$$

$$e_2' = e_2 - (1 - cos\theta)cos\beta(cos\beta e_2 - sin\beta e_1) - sin\theta cos\beta e_3, \tag{3.24}$$

$$e_3' = cos\theta e_3 + sin\theta(cos\beta e_2 - sin\beta e_1 \tag{3.25}$$

where β solves the equation

$$\frac{d\beta}{ds} = C sin\beta - \kappa, \tag{3.26}$$

and $tan(\frac{\theta}{2}) = \frac{C}{\tau}$, has torion $\tau' = \tau$, and curvature

$$\kappa' = \kappa - 2C sin\beta. \tag{3.27}$$

From the results of the previous section, this clearly implies that the above transformation constitutes an auto-Bäcklund transformation for QCD_2.

Let us illustrate this Bäcklund transformation with an example. First, lets find the Bäcklund transformation of the "trivial" solution of (3.9) - (3.11) , i.e. the solution with $\kappa = 0$ and $\tau = \sqrt{2}g^2x^+$. A suitable Frenet frame for this curve is given by

[5] i.e. a surface with constant negative Gauss curvature.

$$e_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} sin(\sqrt{2}g^2x^+s) \\ cos(\sqrt{2}g^2x^+s) \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} cos(\sqrt{2}g^2x^+s) \\ -sin(\sqrt{2}g^2x^+s) \\ 0 \end{bmatrix}. \tag{3.28}$$

For $\kappa = 0$,(3.26) has the solution

$$\beta = -2arctan(e^{Cs+\alpha}) \tag{3.29}$$

which corresponds to a curve with curvature $\kappa' = -2Csin\beta = 2asech(Cs + \alpha)$. The tangent to this Bäcklund transformed curve follows directly from (3.23)

$$e_1 = \begin{bmatrix} \frac{-2a^2}{a^2+2g^4x^{+2}} sech(as)tanh(as)sin(\sqrt{2}g^2x^+s) - \frac{2\sqrt{2}ag^2x^+}{a^2+2g^4x^{+2}} cos(\sqrt{2}g^2x^+s) \\ \frac{-2a^2}{a^2+2g^4x^{+2}} sech(as)tanh(as)cos(\sqrt{2}g^2x^+s) + \frac{2\sqrt{2}ag^2x^+}{a^2+2g^4x^{+2}} sin(\sqrt{2}g^2x^+s) \\ 1 - \frac{2a}{a^2+2g^4x^{+2}} sech^2(as) \end{bmatrix}. \tag{3.30}$$

This is nothing but the soliton solution (3.21) we found earlier.

It is clear that in order to implement the Bäcklund transformation we need to solve the associated equation (3.26) , which is difficult for non trivial curvatures. In the remainder of this section, we describe how to solve (3.26) for any curve whose Frenet frame is explicitly known and can be continued analytically as a function of τ, with fixed curvature.

First, setting $y = tan\frac{\beta}{2}$, (3.26) becomes

$$\frac{dy}{ds} = ay - \frac{\kappa}{2}(1 + y^2). \tag{3.31}$$

Now, if we analytically continue the (real) torsion to $\tau = -ia$, then we find that we can identify $y = -z_l$ where z_l is a Darboux coordinate (3.12) satisfying the Ricatti equation (3.13) . This provides a direct way of obtaining β.

Lets illustrate this last point with an example. From the Frenet frame (3.28) we construct the Darboux coordinate $z_2 = cos(\tau x) + isin(\tau x)$. Evaluating this at $\tau = -ia$ yeilds $y = -e^{ax}$, which implies $\beta = -2arctan(e^{ax})$ in agreement with (3.29) .

3.4. Fermion Number.

Our classical solutions should correspond to solutions with a well defined fermion number. Thus, we should be able to identify an integer valued charge associated with each of the above classical solutions. In this section we show that this fermion number arises in a very natural way.

First, note that from (2.3), it is clear that the quark field ψ_a is an eigenvector of $\partial_-^2\sigma$ with eigenvalue equal to the fermion number. Since $\partial_-^2\sigma$ is a two dimensional traceless Hermittian matrix, it has two real eigenvalues of opposite sign. The positive eigenvalue corresponds to the fermion number density. Thus, we may write

$$\psi_a^\dagger\psi_a = \sqrt{\frac{1}{2}Tr(\partial_-^2\sigma)^2}. \tag{3.32}$$

In the curve language, this is simply the curvature. Thus, the total fermion number should be proportional to the integral over arclength of the curvature of the solution.

Now, consider the set of solutions that arise as Bäcklund transformations of the solution with $\kappa = 0$. These solutions have curvature

$$\kappa = -2Csin(\beta), \tag{3.33}$$

where

$$\frac{d\beta}{ds} = Csin(\beta). \tag{3.34}$$

To obtain a soliton solution, we seek solutions which have the property $\frac{d\beta}{ds} \to 0$ as $s \to \pm\infty$. From the above equation, this tells us that $\beta \to n\pi$ as $s \to \pm\infty$ for any integer n . Thus, the total fermion number is proportional to

$$\int ds\kappa(s) = -\int ds2Csin(\beta) = -2\int ds\frac{d\beta}{ds} = -2\beta|_{-\infty}^{\infty} = -2(n_1 - n_2)\pi. \tag{3.35}$$

Thus, the fermion number for these solutions are an integer times 2π . Now, consider the solution obtained by Bäcklund transformation of the above solution. In this case the curvature is given in (3.27) where the dynamics of β is now (3.26). The boundary conditions for β are unchanged, so that the above argument again leads to the conclusion that the total fermion number is an integer multiplied by 2π. Indeed, any solution which is obtained from a sequence of Bäcklund transformations of the $\kappa = 0$ solution, have a fermion

number equal to 2π multiplied by an integer. The above arguments can be generalized to solutions obtained using the auto-Bäcklund transformation on a curve with an arbitrary initial curvature. In this case, the fermion number of the solutions obtained through the Bäcklund transformation will differ from the fermion number of the original curve by $2\pi n$ where n is an arbitrary integer. There is however no restriction on the fermion number of the initial curve. Note that the results discussed in this section depend on the fact that x^- is identified with the arclength along the curve.

3.5. Closed Curves.

In this section, we consider the following differential-geometric problem: If a curve in R^3 is closed, then the curvature and torsion are necessarily periodic. However, given a periodic curvature and a periodic torsion, we do not in general obtain a closed in curve. In this section, we will show that the extra constraints that must be placed on the curvature and the torsion so as to obtain a closed curve, are naturally expressed in terms of the spectral problem for QCD_2. More precisely, the requirement that a curve is periodic can be expressed as a constraint on the spectral curve of classical chiral $SU(2)$ QCD_2. It is interesting to study this curve, since spectral curves associated with other $(1+1)$ dimensional models have been used to derive the Seiberg-Witten differential (and therefore also the exact mass-charge relationship of the BPS saturated states appearing in the low energy dynamics [20]). The observations made in this section are based on the recent proof given in [21] where Riemann surfaces corresponding to periodic curves are identified.

As a start, consider the zero curvature condition given in (3.18) and (3.19) . Since the zero curvature condition is invariant under gauge transformations, we can equivalently consider the spectral problem[6]

$$\begin{aligned} U \to \tilde{U} &= g^{-1}Ug - g^{-1}\partial_s g, \\ V \to \tilde{V} &= g^{-1}Vg - g^{-1}\partial_{x^+} g, \end{aligned} \tag{3.36}$$

where the connection U, V are defined in (3.18) and (3.19) , and

$$g = \begin{bmatrix} exp(\frac{i}{2}\int_0^{x^-} dx'\tau(x')) & 0 \\ 0 & exp(-\frac{i}{2}\int_0^{x^-} dx'\tau(x')) \end{bmatrix}. \tag{3.37}$$

[6] In what follows, we have normalized the length of the closed curve (given by traversing the curve once) to 2π.

Although this gauge transformation changes the connections U and V, it leaves the integrability condition unchanged. Concentrate on the problem

$$L(\lambda)F(s,\lambda)=0=\Big(\frac{d}{ds}-U\Big)F(s,\lambda),\quad L(\lambda)=\frac{d}{ds}-\begin{bmatrix}-\frac{1}{2}i\lambda & \frac{1}{2}iq(s)\\ \frac{1}{2}i\bar{q}(s) & \frac{1}{2}i\lambda\end{bmatrix}. \tag{3.38}$$

The crucial elements of the above problem are that the magnitude of $q(s)$ is related to the curvature of the curve, and that the derivative with respect to arclength of the phase of $q(s)$ is related to the torsion. It is this property of QCD_2 that makes it possible to consider the closed curve problem. The link just described, between a curve and a function $q(s)$ is known as the Hasimoto map[17] . In what follows, we will need to make use of the shift operator, defined by the action

$$f(s)\to f(s+2\pi). \tag{3.39}$$

Here, since we are looking for a closed curve, both the torsion and curvature are periodic. It thus makes to sense to think that our quarks will be moving (at fixed x^+) in a periodic potential. In analogy to the treatment of waves moving through a periodic potential (lattice) we define Bloch functions as the functions which are eigenfunctions of both the shift operator and $L(\lambda)$

$$\begin{gathered}\psi^{(1)}=\begin{bmatrix}\psi_1^{(1)}(s,\lambda)\\ \psi_2^{(1)}(s,\lambda)\end{bmatrix},\quad \psi^{(2)}=\begin{bmatrix}\psi_1^{(2)}(s,\lambda)\\ \psi_2^{(2)}(s,\lambda)\end{bmatrix},\\ L(\lambda)\psi^{(1)}(s,\lambda)=0,\quad \psi^{(1)}(s+2\pi,\lambda)=\omega^{(1)}(\lambda)\psi^{(1)}(s,\lambda),\\ L(\lambda)\psi^{(2)}(s,\lambda)=0,\quad \psi^{(2)}(s+2\pi,\lambda)=\omega^{(2)}(\lambda)\psi^{(2)}(s,\lambda).\end{gathered} \tag{3.40}$$

Since $U(s,\lambda)$ is traceless, the normalization of the Bloch functions does not change with arclength s, so that we must have

$$\omega^{(1)}(\lambda)\omega^{(1)*}(\lambda)=\omega^{(1)}(\lambda)\omega^{(2)*}(\lambda)=1. \tag{3.41}$$

A point in the comple plane λ belongs to the spectrum of $L(\lambda)$ if and only if $|\omega^{(1)}(\lambda)|=1$. Thus, we have obtained our first characetrization of our spectral curve. Now, we will use the concept of a transition matrix, which is introduced in the next section. The reader is asked to consult that section for details.[7] For a generic complex λ, the transition matrix

[7] We trust that this will not confuse the reader.

$T(\lambda)$ will have two eigenvalues $\omega^{(1)}(\lambda)$ and $\omega^{(2)}(\lambda)$ and a pair of corresponding Bloch functions. In terms of these two functions, we introduce a hyperelliptic Riemann surface[8] Y such that $\omega^{(1)}(\lambda) = \omega(\mu_1)$ and $\omega^{(2)}(\lambda) = \omega(\mu_2)$ where μ_1 and μ_2 are the pre-images of the point λ under the projection $Y \to C$. Note that any hyperelliptic Riemann surface obviously has a natural holomorphic involution which amounts to transposing the two sheets. The function

$$p(\mu) = \frac{1}{2\pi i} ln\omega(\mu) \tag{3.42}$$

is called the quasimomentum function. In what follows, the quasimomentum differential

$$dp(\mu) = \frac{dp(\mu)}{d\mu} d\mu \tag{3.43}$$

will play an important role.

The structure of the Riemann surface Y (called the Bloch variety) has been studied in detail in [22] . We will need the following definitions: A complex point λ is called regular if $\omega^{(1)}(\lambda) \neq \omega^{(2)}(\lambda)$, and irregular if $\omega^{(1)}(\lambda) = \omega^{(2)}(\lambda)$. There are three types of irregular points (1) Branch points, (2) Non-removable double points and (3) Removable double points. An irregular point λ_0 is called a branch point if in going around this point we move from one sheet of Y to another (i.e. the monodromy around this point is non-trivial). If the monodromy around an irregular point is trivial, then this point is a double point. If the Bloch functions are equal at the double point, then it is non-removable; if the Bloch functions are not equal at the double point, then the double point is removable.

One last piece of notation is needed now. Introduce the new $SU(2)$ valued frame $\sigma_i \hat{E}^i_n$, related to the Frenet frame by

$$\begin{aligned} \hat{E}_1 &= e_1, \\ \hat{E}_2 &= cos(\theta) e_2 - sin(\theta) e_3, \\ \hat{E}_3 &= sin(\theta) e_2 - cos(\theta) e_3, \\ \theta &= \int^s \kappa(s') ds'. \end{aligned} \tag{3.44}$$

[8] A Riemann surface is called hyperelliptic if it is a two sheeted ramified covering of the Riemann sphere.

The $SU(2)$ valued coordinate of the curve is now given by

$$\hat{\Gamma}(s) = \hat{\Gamma}(0) + \int_0^s ds' \hat{E}_1(s'). \tag{3.45}$$

We are now ready to state the theorem that provides the restriction on the spectral curve: Let $q(s)$ be a complex valued smooth periodic function of one variable s, $q(s) \neq 0$ for all s, $q(s+2\pi) = q(s)$, Λ_0 an arbitrary real number and $\hat{\Gamma}(s)$ the corresponding curve constructed from the curvature and torsion specified in $q(s)$. Then, (1) the matrix $\hat{E}_1(s)$ is periodic with period 2π, if and only if Λ_0 is a double point of the Bloch variety Y and (2) the function $\hat{\Gamma}(s)$ is periodic with period 2π if and only if Λ_0 is a double point of Y and $dp(\mu_1) = 0 = dp(\mu_2)$ where μ_1 and μ_2 are the preimages of Λ_0 under the projection $Y \to C$. We are content with the statement of the theorem, and refer the reader to [21] for a proof.

To summarize, we have managed to show that the spectral problem in QCD_2 can be used to define Bloch functions for periodic potentials. The requirement that these periodic potentials actually correspond to closed curves is then easily expressed as a condition on the quasimomentum differential.

3.6. A Nonlinear Superposition Principle.

The usual approach to massive QCD_2 is to begin by performing a non-Abelian bosonization. Then, classical solutions of the bosonic action are found and a semi classical quantization may be carried out. The static classical solutions of these bosonic equations satisfy the Sine-Gordon equation. This allows the construction of multi-soliton solutions using the well known nonlinear superposition associated with the Sine-Gordon equation [23] . In this section, we show that the nonlinear superposition principle also plays a role for the chiral theory. This explicitly demonstrates the existence of multi soliton solutions, which is another signal of the theories classical integrability.

Begin by considering the spatial part of the Bäcklund transformation for the Sine-Gordon equation

$$\frac{\partial(\tilde{\varphi} - \varphi)}{\partial s} = C sin(\tilde{\varphi} + \varphi), \tag{3.46}$$

where φ is the one soliton solution and hence solves

$$\frac{\partial \varphi}{\partial s} = sin \varphi. \tag{3.47}$$

From this last equation, it is clear that the single soliton solution corresponds to a curve with curvature $-2sin\varphi$. Now, noting that

$$\frac{\partial(\varphi+\tilde{\varphi})}{\partial s}=Csin(\varphi+\tilde{\varphi})+2\frac{\partial\varphi}{\partial s}=Csin(\varphi+\tilde{\varphi})+2Csin(\varphi) \tag{3.48}$$

we see that we may identify $\beta=\varphi+\tilde{\varphi}$ to obtain a Bäcklund transformation to a curve with curvature $-2sin\varphi-2sin\beta$. Now, lets compute the three soliton solution $\bar{\varphi}$ from the two soliton solution $\tilde{\varphi}$

$$\frac{\partial(\bar{\varphi}-\tilde{\varphi})}{\partial s}=Csin(\bar{\varphi}+\tilde{\varphi}). \tag{3.49}$$

Rewriting this equation as

$$\frac{\partial(\bar{\varphi}+\tilde{\varphi})}{\partial s}=Csin(\bar{\varphi}+\tilde{\varphi})+2\frac{\partial\tilde{\varphi}}{\partial s}=Csin(\bar{\varphi}+\tilde{\varphi})+2(Csin(\varphi+\tilde{\varphi})+sin(\varphi)), \tag{3.50}$$

so that identifying $\beta=\bar{\varphi}+\tilde{\varphi}$ provides a Bäcklund transformation to a curve of curvature $-2sin(\varphi+\bar{\varphi}+\tilde{\varphi})-2sin(\varphi+\tilde{\varphi})-2sin(\varphi)$. By now, the generalization to the n soliton solution is obvious.

From the nonlinear superposition for the Sine-Gordon equation, we obtain the algebraic formula

$$\beta=2arctan\Big[\frac{c_1+c_2}{c_1-c_2}tan\Big(\frac{\phi_n^1-\phi_n^2}{2}\Big)\Big]+\phi_n^1+\phi_{n-1}, \tag{3.51}$$

for the Bäcklund transformation contructed from the $n+1$ soliton solution of the Sine-Gordon equation. In this last formula, ϕ_n^1 is the n soliton solution (of the Sine-Gordon equation)constructed from the $n-1$ soliton solution ϕ_{n-1} (of the Sine-Gordon equation) with parameter c_1, ϕ_n^2 is the n soliton solution (of the Sine-Gordon equation) constructed from ϕ_{n-1} with parameter c_2. Thus, the nonlinear superposition principle for the Sine-Gordon equation can be used to construct the multi soliton solutions of chiral QCD_2.

4. Dynamical Analysis using the Inverse Scattering Method.

In this section, we present a systematic inverse scattering treatment of the classical equations of motion for the gauge group $SU(2)$. We reproduce the single soliton solution described in the previous section, clearly demonstrating the validity of the analysis. However, in contrast to the previous section, all results in this section are easily generalized to $SU(N)$.

4.1. The direct scattering problem.

In this section, we study the linear problem (2.22) derived in section 2.3. The eigenvalue problem of the Lax operator reads

$$\partial_- \phi_a = \lambda \partial_- \sigma_{ab} \phi_b. \tag{4.1}$$

From the time dependance of $\partial_- \sigma$, (2.14) , it is clear that requiring that $\partial_+ \partial_- \sigma \to 0$ as $|x^-| \to \infty$ implies that the commutator of σ and $\partial_- \sigma$ must vanish as $|x^-| \to \infty$. There are two ways in which this commutator may vanish. First, the field $\partial_- \sigma$ itself vanishes. Since we are working in terms of the arclength variable s, we have

$$(e_1)_i (e_1)_i = \frac{1}{2} Tr(\partial_- \sigma^2) = 1, \tag{4.2}$$

so that $\partial_- \sigma \neq 0$ as $|x^-| \to \infty$. The second possibility is that σ becomes proportional to a single Pauli matrix. Without loss of generality (thanks to global color rotation invariance), we may take σ to be proportional to σ_z. The requirement (4.2) forces $\partial_- \sigma \to \sigma_z$ as $|x^-| \to \infty$, which fixes the relevant boundary condition for (4.1) . This linear problem has been studied by Takhtajan [24] (see also Fogedby [25]) in connection with the classical Heisenberg spin chain. In this section we will review the results of his work relevant for the present analysis.

The two Jost solutions $F(x, \lambda)$ and $G(x, \lambda)$ defined by the boundary conditions

$$F(x, \lambda) \to exp(-i\lambda \sigma_3 x) \tag{4.3}$$

as $x \to \infty$ and

$$G(x, \lambda) \to exp(-i\lambda \sigma_3 x) \tag{4.4}$$

as $x \to -\infty$, have the integral representations

$$F(x, \lambda) = exp(-i\lambda \sigma_3 x) + \lambda \int_x^{\infty} K(x, y) exp(-i\lambda \sigma_3 y) dy \tag{4.5}$$

$$G(x, \lambda) = exp(-i\lambda \sigma_3 x) + \lambda \int_{-\infty}^{x} N(x, y) exp(-i\lambda \sigma_3 y) dy. \tag{4.6}$$

The kernels $K(x, y)$ and $N(x, y)$ are independant of the eigenvalue λ and solve the Goursat problem

$$\frac{\partial K(x,y)}{\partial x}\sigma_3 + \partial_-\sigma(x)\frac{\partial K(x,y)}{\partial y} = 0 \quad (x \leq y), \tag{4.7}$$

$$\frac{\partial N(x,y)}{\partial x}\sigma_3 + \partial_-\sigma(x)\frac{\partial N(x,y)}{\partial y} = 0 \quad (x \geq y), \tag{4.8}$$

with the boundary conditions

$$\partial_-\sigma(x) - \sigma_3 + iK(x,x) - i\partial_-\sigma(x)K(x,x)\sigma_3 = 0, \tag{4.9}$$

$$\partial_-\sigma(x) - \sigma_3 + iN(x,x) - i\partial_-\sigma(x)N(x,x)\sigma_3 = 0. \tag{4.10}$$

These last two boundary conditions imply that

$$\partial_-\sigma(x) = (iK(x,x) - \sigma_3)\sigma_3(iK(x,x) - \sigma_3)^{-1}, \tag{4.11}$$

and

$$\partial_-\sigma(x) = (iN(x,x) - \sigma_3)\sigma_3(iN(x,x) - \sigma_3)^{-1}. \tag{4.12}$$

The linear system (4.1) has the important property that if Ψ_1 and Ψ_2 are solutions, then $\Psi_2 = \Psi_1 A$, with A a constant.[9] This allows us to write

$$G(x,\lambda) = F(x,\lambda)T(\lambda), \tag{4.13}$$

where $T(\lambda)$, is the transition matrix. If Ψ is a solution of (4.1) , then writing $det(\Psi)$ as $exp(Trln\Psi)$ we find, upon using $Tr(\partial_-\sigma) = 0$, that $det(\Psi)$ is independant of x^-. Also, since the Pauli matrices have the property that $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$, it is clear that if Ψ is a solution of (4.1) for a spectral parameter λ, then $\sigma_2\Psi^*\sigma_2$ is a solution for λ^*. Using these properties of the solutions to (4.1) , it is not difficult to show that the transition matrix has the form

$$T(\lambda) = \begin{bmatrix} a(\lambda) & -(b(\lambda^*)^* \\ b(\lambda) & (a(\lambda^*)^* \end{bmatrix}, \tag{4.14}$$

where

[9] This property is easily proved by differentiating $\Psi_2^{-1}\Psi_1$ with respect to x^-, and using (4.1) .

$$a(\lambda)a(\lambda^*)^* + b(\lambda)b(\lambda^*)^* = 1. \tag{4.15}$$

The scattering data for the operator $\partial_-\sigma$ is the set $s = \{a(\lambda), b(\lambda); \xi_j, m_j,\ Im(\xi_j > 0, j = 1, ..., n\}$. $a(\lambda)$ can be analytically continued to the half plane $Im(\lambda) > 0$, ξ_j are the zeroes of $a(\lambda)$, and $m_j = \frac{-ib(\xi_j)}{\partial_- a(\xi_j)}$. This last expression is written with the implicit assumption that we consider only simple zeroes ξ_j.

4.2. Time dependance of the scattering data.

The transition matrix is independant of x^-. This allows us to extract the x^+ dependance of the transition matrix, in the limit $x^- \to \infty$. Now, recalling $\lambda = i\frac{g^2}{\sqrt{2}}x^+ + f(x^-)$, and using the boundary condition of the σ field, we find

$$\frac{i\sqrt{2}}{g^2}\partial_+ G = x^-\sigma_3 G. \tag{4.16}$$

Alternativly, we may write $G = FT$, and use the boundary condition for F, which implies

$$\frac{i\sqrt{2}}{g^2}\partial_+ G = x^-\sigma_3 e^{-i\lambda\sigma_3 x^-} + e^{-i\lambda\sigma_3 x^-}\frac{i\sqrt{2}}{g^2}\partial_+ T. \tag{4.17}$$

Comparing these last two equations, we find that

$$\frac{dT}{dx^+} = 0 = \frac{\partial T}{\partial x^+} + \frac{\partial\lambda}{\partial x^+}\frac{\partial T}{\partial\lambda}. \tag{4.18}$$

This last equation has the solution

$$T = T(\lambda - \frac{g^2}{\sqrt{2}}x^+). \tag{4.19}$$

This is a little unusual. For most models (for example Sine Gordon model, Heisenberg spin chain, nonlinear Schrödinger model) the spectral parameter is a constant and the scattering evolve in time. Here we have a linear time dependance for the spectral parameter and a set of scattering data which do not evolve in time!

4.3. The inverse scattering problem.

Takhtajan has shown the kernel $K(x,y;t)$ may be reconstructed from the time dependant scattering data, using the Gelfand-Levitan-Marchenko equation[10]

$$K(x^-,y^-;x^+)+\Phi_1(x^-+y^-;x^+)+\int_{x^-}^{\infty}K(x^-,z^-;x^+)\Phi_2(z^-+y^-;x^+)dz^-=0 \quad (4.20)$$

where $x^- \leq y^-$,

$$\Phi_1=\begin{bmatrix}0 & -\Lambda^* \\ \Lambda & 0\end{bmatrix}, \quad (4.21)$$

$$\Phi_2=-i\begin{bmatrix}0 & \partial_-\Lambda^* \\ \partial_-\Lambda & 0\end{bmatrix} \quad (4.22)$$

and

$$\Lambda(x^-;x^+)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\lambda,x^+)}{\lambda a(\lambda,x^+)}e^{i\lambda x^-}d\lambda+\sum_{j=1}^{N}\frac{m_j(x^+)}{\xi_j}e^{i\xi_j x^-}. \quad (4.23)$$

Having solved (4.20) for a set of scattering data, $\partial_-\sigma$ is then reconstructed using (4.11) .

4.4. Soliton solutions: Reflectionless Potentials.

If the reflection coefficient $r(\lambda)=\frac{b(\lambda)}{a(\lambda)}$ is zero, then the kernel and inhomogeneous terms in the Gelfand-Levitan-Marchenko equation are reduced to finite sums over the discrete spectrum. In particular,

$$\Lambda(x^-;x^+)=\sum_{j=1}^{N}\frac{m_j(x^+)}{\xi_j}e^{i\xi_j x^-}. \quad (4.24)$$

Now, making the ansatz

$$K=M(x^-,x^+)\begin{bmatrix}\sum_j a_j e^{i\xi_j z^-} & 0 \\ 0 & \sum_j a_j^* e^{-i\xi_j z^-}\end{bmatrix} \quad (4.25)$$

[10] Fogedby[25] has given a clear derivation of this equation in the context of the classical Heisenberg spin chain.

we find that (4.20) reduces to a system of linear algebraic equations

$$
\begin{aligned}
M(x^-,x^+)&\begin{bmatrix} \sum_j a_j e^{i\xi_j y^-} & -\sum_{jk} \frac{a_k m_j^* e^{-ix_-(\xi_j^*-\xi_k)-i\xi_j^* y^-}}{i(\xi_j^*-\xi_k)} \\ -\sum_{jk} \frac{a_k^* m_j e^{ix^-(\xi j-\xi_k^*)+i\xi_j y^-}}{i(\xi_j-\xi_k^*)} & \sum_j a_j^* e^{-i\xi_j^* y^-} \end{bmatrix} \\
&+\begin{bmatrix} 0 & -\sum_j \frac{m_j^*}{\xi_j^*} e^{-i\xi_j^*(x^-+y^-)} \\ \sum_j \frac{m_j}{\xi_j} e^{i\xi_j(x^-+y^-)} & 0 \end{bmatrix} = 0.
\end{aligned} \tag{4.26}
$$

which may be solved explicitly. In this way, we obtain the N soliton solution for QCD_2.

For the one soliton solution, we take a single bound state eigenvalue ξ. In this case, the kernel which solves the GLM equation is given by

$$
\begin{aligned}
K(x^-,z^-) = &\frac{|m|^2(\xi-\xi^*)^2}{|a|^2|\xi|^2[(\xi-\xi^*)^2 e^{-ix^-(\xi-\xi^*)} - |m|^2 e^{-ix^-(\xi-\xi^*)}]}\times \\
&\begin{bmatrix} \frac{|a|^2\xi}{i(\xi-\xi^*)} e^{-i(x^-\xi^*-\xi z^-} & \frac{\xi|a|^2}{m} e^{-i(\xi x^-+\xi^* z^-)} \\ -\frac{|a|^2\xi^*}{m^*} e^{i(\xi^* x^-+\xi z^-)} & \frac{|a|^2\xi^*}{i(\xi-\xi^*)} e^{i(x^-\xi-\xi^* z^-)} \end{bmatrix}
\end{aligned} \tag{4.27}
$$

Inserting this kernel into (4.11) yields

$$
\partial_- f_3 = 1 - \frac{(2Im(\xi))^2}{(2Im(\xi))^2+(2Re(\xi))^2} sech^2\big[2x^- Im(\xi) + ln|\frac{2Im(\xi)}{|m|}|\big], \tag{4.28}
$$

and

$$
\partial_- f_1 + i\partial_- f_2 = \sqrt{1-(\partial_- f_3)^2} e^{i\phi}, \tag{4.29}
$$

where

$$
\begin{aligned}
\phi = arg(m) + 2Re(\xi)x^- + arctan\Big[&\frac{Im(\xi)}{|\xi|}\sqrt{\frac{|\xi|^2}{|\xi|^2-(Im(\xi))^2}}\times \\
&tanh\Big(2x^- Im(\xi) + ln|\frac{2Im(\xi)}{|m|}|\Big)\Big].
\end{aligned} \tag{4.30}
$$

Now, from the time dependance of the spectral parameter, it is clear that m is a complex constant, and that

$$
\xi = \frac{g^2}{\sqrt{2}} x^+ + \bar{x}^+ + i\frac{a}{2}, \tag{4.31}
$$

where $\bar{x}^+$ and a are real constants. Upon inserting this into (4.28) and (4.29) , we find that we are reproducing the one soliton solution discussed in section 3.2.

References

[1] E. Abdalla and M.C.B. Abdalla, Phys. Lett. B337 (1994) 347; E. Abdalla and M.C. Abdalla, Int. Jour. Mod. Phys. A10, (1995) 1611; E. Abdalla and M.C. Abdalla, Phys. Rev. D52 (1995) 6660; E. Abdalla and M.C. Abdalla, Phys. Rept. 265 (1996) 253.

[2] D.J. Gross, I.R. Klebanov, A.V. Matytsin and A.V. Smilga, Nucl. Phys. B461 (1996) 109.

[3] H.B. Thacker, Rev. Mod. Phys. 53 (1981) 253; N. Andrei, K. Furuya and J.H. Lowenstein, Rev. Mod. Phys. 55 (1983) 331; A.Zamolodchikov and A.Zamolodchikov, Ann. Phys. 120 (1979) 253; L. Faddeev, in Les Houches Summer School in Theoretical Physics, Session 19:Recent Advances in Field Theory and Statistical Mechanics, Les Houches, France; Aug2-Sep10, 1982.

[4] C.G. Callan Jr., N. Coote and D.J. Gross, Phys. Rev. D13 (1976) 1649.

[5] W.Krauth and M.Staudacher, Phys. Lett. B388 (1996) 808.

[6] G.'t Hooft, Nucl. Phys. B75 (1974) 461.

[7] E.Abdalla and R.Mohayaee,*'Quasi-integrability and two-dimensional QCD'*, CERN Preprint CERN-TH/96-253, hep-th/9610059.

[8] E. Abdalla, R. Mohayaee and A. Zadra, *'Screening in Two-dimensional QCD'*, Preprint IC/96/51, hep-th 9604063; R. Mohayee, *'The phases of two dimensional QED and QCD'*, hep-th 9705243; Y. Frishman and J. Sonnenschein, Nucl. Phys. B496 (1997) 285; J. Sonnenschein, *'More on Screening and Confinement in 2D QCD'*, hep-th 9707031; A. Armoni and J. Sonnenschein, *'Screening and Confinement in large N_f QCD_2 and in $N=1$ SYM_2'*, Preprint TAUP-2412-97, hep-th 9703114.

[9] R. de Mello Koch and J.P. Rodrigues, *'Classical integrability of chiral QCD_2 and classical curves'*, CNLS Preprint CNLS-97-01, hep-th/9701138, to be published in Mod. Phys. Lett. A.

[10] E. Cremmer, H. Lü, C.N. Pope and K.S. Stelle, *'Spectrum-generating Symmetries from BPS Solitons'*, Preprint LTP TAMU-29/97, Imperial/TP/96-97/54, LPTENS-97/34, SISSARef 92/97/EP, hep-th 9707207.

[11] R. de Mello Koch and J.P. Rodrigues, Phys. Rev. D54, 7794 (1996).

[12] N. Papanicoloau, Ann. Phys. 136 (1981) 210.

[13] S.J.Orfanidis, Phys. Rev. D14 (1976) 472.

[14] R. de Mello Koch and J.P. Rodrigues, in preparation.

[15] A. Jevicki and N. Papanicoloau, Ann. Phys. 120 (1979) 107.

[16] L. Faddeev, in Les Houches Summer School in Theoretical Physics, Session 19:Recent Advances in Field Theory and Statistical Mechanics, Les Houches, France, Aug2-Sep10, 1982.

[17] Hasimoto, J. Fluid Mech. 51 (1972) 477.

[18] G.L. Lamb, Jour. Mth. Phys. 18, (1977) 1654.

[19] A.Calini and T.Ivey, *'Backlund transformations and knots of constant torsion'*, Preprint dg-ga/9608001.

[20] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299.

[21] P.G. Grinevich and M.U. Schmidt, *'Closed Curves in R^3: a characterization in terms of curvature and torsion, the Hasimoto map and periodic solutions to the Filament equation'*, dg-ga 9703020.

[22] M.U. Schmidt, Memoirs of the Amer. Math. Soc. (1996) 551.

[23] C.Rogers in "Soliton Theory: a survey of results", ed. A.P. Fordy, St. Martin's Press, 1990.

[24] L. Takhtajan, Phys. Lett. A64 (1977) 235.

[25] H.Fogedby, J. Phys. A Math. Gen. 13 (1980) 1467.